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LETTER TO THE EDITOR

Rotating Rayleigh–Benard convection with modulation

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Abstract. We consider rotating Rayleigh–Benard convection with modulated rotation speed. Galerkin truncation under realistic, i.e. experimentally realisable, boundary conditions is carried out. The threshold of convection can be raised or lowered depending on the Prandtl number and rotation speed.

The effect of modulation of the control parameter on the onset of hydrodynamic instability has been a subject of continuing interest. Recently, the effect of temperature modulation on the Rayleigh–Benard instability and the effect of modulation of the rotation speed in the Taylor–Couette instability has been probed extensively both theoretically and experimentally (Ahlers *et al* 1985, Meyer *et al* 1988, Niemela and Donnelly 1986, Kumar *et al* 1986, Walsh and Donnelly 1988). For Rayleigh–Benard convection the temperature modulation is supposed to stabilise the conduction state. Complications set in, however, since the temperature modulation breaks the reflection symmetry about the mid-plane and hexagons, rather than cylinders, constitute the convection plan-form immediately above the threshold. For the Rayleigh–Benard problem with rotation, the above problem can be avoided if the rotation speed is modulated. This leads to a cleaner problem for the study of the effect of modulation on the threshold. For high rotation rates, specifically rates higher than the critical values for the onset of the Küppers–Lortz (KL) instability (Küppers and Lortz 1969), the unmodulated state makes a direct transition to a weakly turbulent state and hence the study of modulation effects become subtle from the theoretical standpoint.

In this letter, we set up a Galerkin truncation of the hydrodynamic equations to study the effect of modulation on the convection threshold. We find that:

(i) for low Prandtl numbers, the conduction state is stabilised for all rotation rates below the onset of KL instability;

(ii) for high Prandtl numbers, the conduction state can be stabilised or destabilised depending upon the rotation rate, which is still below the onset of KL instability;

(iii) for rotation rates above the threshold for KL instability, the Galerkin model has to be supplemented by additional information about the instability, and in this regime the effect of the modulation is to destabilise the conduction state.

The hydrodynamic equations for the problem are the Navier–Stokes equation with the Coriolis and centrifugal forces included and the heat conduction equation which can be written as (Chandrasekhar 1961)

$$\frac{\partial \boldsymbol{v}}{\partial t}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}=-\frac{\nabla p}{\rho}+\nu\nabla^2\boldsymbol{v}+\boldsymbol{g}+2\boldsymbol{\Omega}(t)\times\boldsymbol{v}+\tfrac{1}{2}\nabla(|\boldsymbol{\Omega}(t)\times\boldsymbol{r}|^2) \tag{1a}$$

$$\frac{\partial T}{\partial t}+(\boldsymbol{v}\cdot\nabla)T=\lambda\nabla^2 T \tag{1b}$$

with

$$\Omega(t)=\Omega_0(1+\varepsilon\cos\omega t). \tag{2}$$

The fluid is contained between two infinite parallel plates a distance d apart in the z direction. The boundary condition on $\boldsymbol{v}$ is the realistic one of no slip. This means that all components of $\boldsymbol{v}$ vanish at the boundaries, as does the z component of the vorticity. We work within the Boussinesq approximation.

The Galerkin truncation modes (Bhattacharjee and McKane 1988) for W (the z component of the velocity), ζ (the z component of the vorticity) and θ (temperature deviation from the conduction state profile) are taken as

$$W(x,z,t)=a(t)(\cos ax)(z^2-\tfrac{1}{4})^2 \tag{3a}$$

$$\zeta(x,z,t)=b(t)(\cos ax)z(z^2-\tfrac{1}{4}) \tag{3b}$$

$$\theta(x,z,t)=c(t)(\cos ax)(z^2-\tfrac{1}{4})+d(t)(z^2-\tfrac{1}{4})^2. \tag{3c}$$

Using X, G, Y and Z to denote the appropriately scaled forms of a, b, c and d, we are led to the dynamical system

$$\dot{X}=\sigma(-X+Y+\tilde{t}G)+\sigma\varepsilon\tilde{t}\cos(\omega\tau)G \tag{4}$$

$$f_1\dot{G}=-\sigma(42+a^2)G-\sigma\tilde{t}X-\sigma\varepsilon\tilde{t}\cos(\omega\tau)X \tag{5}$$

$$f_2\dot{Y}=-XZ+rX-Y \tag{6}$$

$$\dot{Z}=XY-bZ \tag{7}$$

where

$$r=\frac{189}{196}\frac{Ra^2}{a^2+10}(a^4+24a^2+504)^{-1}$$

$$f_1=(a^4-24a^2+504)/(12+a^2) \qquad f_2=f_1/(10+a^2)$$

$$\tilde{t}^2=12(a^4+24a^2+504)^{-1}\tilde{T} \qquad \tilde{T}=4\Omega^2d^4/\nu^2$$

$$b=12/f_1$$

and time scaling is done by the factor $\sigma d^2/\nu f_1$.

We have checked that the linear stability analysis of the conduction state $X=G=Y=Z=0$ leads to threshold Rayleigh-Benard numbers and critical wavenumbers in excellent agreement with the exact answers. Further, if instead of thermally conducting boundaries, thermally insulating boundaries are used, appropriate changes in the z-dependent functions for $\theta(x,z,t)$ in (3c) lead to excellent answers for the convection threshold once again. Hence, we consider the Galerkin truncation of (4)-(7) as highly accurate and use it to study the effect of modulation.

For the linear stability analysis of the conduction state $X=Y=G=0$, we need the linearised system

$$\dot{X}=\sigma(-X+Y+\tilde{t}G)+\sigma\varepsilon\tilde{t}\cos(\omega\tau)G \tag{8a}$$

$$f_1\dot{G}=-\sigma(42+a^2)G-\sigma tX-\sigma\varepsilon\tilde{t}\cos(\omega\tau)X \tag{8b}$$

$$f_2\dot{Y}=-Y+rX. \tag{8c}$$

In the absence of the modulation, $\varepsilon=0$, the onset of stationary convection occurs at

$$r=r_0=1+\tilde{t}^2/(42+a^2)$$

or

$$R=\frac{196}{189}\left(\frac{(a^2+10)}{a^2}(a^4+24a^2+504)+\frac{12\tilde{T}}{a^2}\frac{a^2+10}{a^2+42}\right).$$

The threshold R is found from the above by minimising with respect to a. The critical a which is thus obtained will be used in (4)-(7) henceforth. To find the shifted threshold we expand

$$r=r_c=r_0+\varepsilon r_1+\varepsilon^2 r_2+\ldots \tag{9a}$$

$$X=X_0+\varepsilon X_1+\varepsilon^2 X_2+\ldots \tag{9b}$$

and similarly for Y and G. One should note that in the expansion for X, G and Y, the zeroth-order terms are time independent for stationary convection, while X_1, $X_2,\ldots$, etc, are the responses at the basic frequency of the modulation and the higher harmonics. Casting (8a)-(8c) in the form

$$L\begin{pmatrix}X\\G\\Y\end{pmatrix}=\varepsilon\sigma\tilde{t}\begin{pmatrix}G\cos\omega\tau\\-X\cos\omega\tau\\0\end{pmatrix}+\varepsilon\begin{pmatrix}0\\0\\r_1+\varepsilon r_2+\ldots\end{pmatrix}X \tag{10}$$

where

$$L=\begin{pmatrix}\partial/\partial\tau+\sigma & -\sigma\tilde{t} & -\sigma\\ \sigma\tilde{t} & f_1\partial/\partial\tau+\sigma(42+a^2) & 0\\ -r_0 & 0 & f_2\partial/\partial\tau+1\end{pmatrix} \tag{11}$$

introducing the expansions of (9a) and (9b) in (10) and equating like powers of ε, we obtain

$$L\begin{pmatrix}X_0\\G_0\\Y_0\end{pmatrix}=0 \tag{12a}$$

$$L\begin{pmatrix}X_1\\G_1\\Y_1\end{pmatrix}=\sigma\tilde{t}\begin{pmatrix}G_0\cos\omega\tau\\-X_0\cos\omega\tau\\0\end{pmatrix}+\begin{pmatrix}0\\0\\r_1\end{pmatrix}X_0 \tag{12b}$$

$$L\begin{pmatrix}X_2\\G_2\\Y_2\end{pmatrix}=\sigma\tilde{t}\begin{pmatrix}G_1\cos\omega\tau\\-X_1\cos\omega\tau\\0\end{pmatrix}+\begin{pmatrix}0\\0\\r_2\end{pmatrix}X_0+\begin{pmatrix}0\\0\\r_1\end{pmatrix}X_1. \tag{12c}$$

We now apply the solvability criterion or the Fredholm alternative on (12b) and (12c), where the operator L is characterised by (11). Straightforward calculation leads to

$$r_1=0 \tag{13}$$

$$r_2=\frac{\sigma}{2}\tilde{t}^2\frac{N}{D} \tag{14}$$

$$N=[f_1+\sigma f_1 f_2+\sigma f_2(42+a^2)]\left(\omega^2 f_2+\frac{4\sigma \tilde{T}^2}{42+a^2}-\frac{\omega^2 \tilde{T}^2 f_1 f_2}{(42+a^2)^2}\right)$$
$$-\left(\frac{3\sigma f_2 t^2}{42+a^2}+\frac{f_1 \tilde{T}^2}{(42+a^2)^2}-\sigma f_2-1\right)$$
$$\times\left((f_2\sigma^2+\sigma)(42+a^2)+\sigma^2\tilde{T}^2 f_2-\frac{\sigma \tilde{T}^2}{42+a^2} f_1-\omega^2\right) \tag{15}$$
$$D=\omega^2[f_1+f_1 f_2\sigma+f_2\sigma(42-a^2)]^2+\left((f_2\sigma^2+\sigma)(42+a^2)+\sigma^2\tilde{T}^2 f_2-\frac{\sigma\tilde{T}^2 f_1}{42+a^2}-\omega^2\right).$$

Typical results are shown in figure 1. It is seen that for low σ, the conduction state is stabilised for all values of Taylor number. For $\sigma \gg 1$, however, the effect is one of stabilisation if the rotation speed is low, but is one of destabilisation if the rotation speed is high. Interestingly enough, it should be remembered that for the modulated Rayleigh-Benard problem the conduction state is always stabilised (Venezian 1969, Ahlers *et al* 1985, Kumar *et al* 1986), while for the modulated Taylor-Couette problem the pure rotational flow is destabilised (Hall 1975, Bhattacharjee *et al* 1986). Experimental results for the rotating Rayleigh-Benard problem with modulated rotation speed should be available shortly (Niemela *et al* 1989).

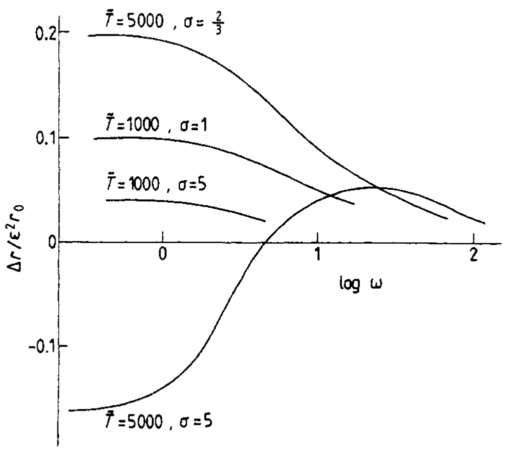

Figure 1. The shift Δr in the threshold Rayleigh number on modulation normalised by the unmodulated threshold r_0 and ε^2, where ε is the amplitude of modulation, plotted against the frequency ω of modulation. $\tilde{T}$ is the Taylor number and σ the Prandtl number. Curves are based on (14)-(16). It should be noted that for very low frequencies the KL instability is to be taken into account and that would always cause a destabilisation at the zero-frequency end.

Finally we address the question of the KL instability. If the rotation speed is greater than the critical value for the onset of KL instability, then at the onset of convection the straight roll system is unstable to a perturbation by a roll system at an angle of approximately 60°. The pattern will consequently be a switching back and forth between three roll systems (Niemella and Donnelly 1987) at 60° to each other (Busse and Heike 1980) and the characteristic timescale for the switching τ_s is quite long. If the modulation frequency is high (i.e. $\omega\tau_s \gg 1$), then the results found above are clearly unaffected.

However, for $\omega\tau_s \leqslant 1$, i.e. for low-frequency modulations, the effect of the switching between different roll systems will be on the same timescale as the modulation, and the modulation will affect the onset of convection. To be more accurate, the convection state produced when the conduction state is destabilised above the KL instability is actually weakly turbulent and the time dependence is characterised by a distribution of frequencies in the low-frequency end. For $\omega\tau_s \leqslant 1$, the modulation frequency lies in this range and a parametric resonance occurs resulting in a lowering of the convection threshold (Bloodworth *et al* 1987).

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